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A group interpretation of particles generated by one dimensional cellular automaton, 54 Wolfram's rule

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Abstract

One dimensional cellular automata are known to be able to present complex behaviors. In some cases, their evolution may be understood as movings, collisions, creations of particles. In the case of the special Wolfram 54 rule, N. Boccara has previously pointed out basic particles. In this paper, we introduce a group which allows to formally study interactions between these particles. Coming back on the complexity of the rule 54, using a new algebraic classification due to I. Rapaport, we give sense to, and prove, the fact that the rule 54 is not simple.

Keywords: cellular automaton, particle, tiling.

Résumé

Les automates cellulaires sont connus pour présenter des comportements complexes. Dans certains cas, il est possible de faire une analogie entre l'évolution de la règle des l'interaction de particules. Dans le cas de la règle 54 de Wolfram, N. Boccara a précédemment mis en évidence les particules élémentaires permettant une telle interprétation. Dans ce papier, nous introduisons un groupe qui permet d'étudier formellement les interactions entre ces particules. Ces résultats nous donnent un nouvel éclairage sur la complexité de la règle 54, en particulier, en utilisant la nouvelle classification algébrique d'I. Rapaport, nous montrons que d'un certain point de vu, la règle 54 n'est pas simple.

Mots-clés: automate cellulaire, particule, pavage.

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Abstract

One dimensional cellular automata are known to be able to present complex behaviors. In some cases, their evolution may be understood as movings, collisions, creations of particles. In the case of the special Wolfram 54 rule, N. Boccara has previously pointed out basic particles. In this paper, we introduce a group which allows to formally study interactions between these particles. Coming back on the complexity of the rule 54, using a new algebraic classification due to I. Rapaport, we give sense to, and prove, the fact that the rule 54 is not simple.

Introduction

A unidimensional cellular automaton (CA) is an infinite succession of identical finite-state machines (indexed by $\mathbb{Z}$) called cells. Each finite-state machine is in a state and these states change simultaneously according to a local transition function: the following state of the machine is related to its own state as well as the states of its two neighbors. A configuration of an automaton is the function which associates to each cell a state. We can thus define a global transition function from the set of all the configurations into itself which associates the following configuration after one step of computation.

An evolution of a unidimensional cellular automaton is usually represented by a space-time diagram. Being given an initial configuration, we represent in $\mathbb{Z} \times \mathbb{N}$ the cellular automaton successive configurations.

Among the unidimensional cellular automata, the most famous is probably the rule 54 according to Wolfram's notation. The main reason of the rule 54 celebrity is Boccara's article which give an interpretation of 54 space-time diagrams by analogy with particle interactions.[2] This rule is in Wolfram's class 3 of chaotic rules but although seems to belong to Wolfram's class 4 which contains cellular automata whose "evolution leads to complex localized structures, sometimes long-lived" and Wolfram speculates "that class 4 cellular automata are characterized by the capability for universal computation" and actually the rule 54 universality is an old conjecture.[7] This rule also appears among the two rules (with 110) that cannot be classify by Cattaneo's method, and thus seems to be one of the two most complex unidimensional cellular automata.[4]

The main result of this paper is to partially explain the particle interactions observed by Boccara. To do this, we will define a group, which is intrinsically linked to the background into which evolve the particles. We will then associate an element of this group to each particle so that the particle interactions have to respect the group structure. This method gives a kind of conservation law on the particle interactions which allow to prove negative results on the rule behavior. Two results will be deduced from this method: the impossibility to annihilate only one of Boccara's particles (g_e) and that a 3 state CA based on 54 is above an infinite number of classes in Rapaport's classification, that we will introduce below.

There are in the literature many attempt of classification of cellular automata. We just mention Wolfram's one which is intuitive but only empirical and Cattaneo's one which classifies the CA according to syntactical properties which is not always intuitive and seems difficult to extend to higher dimensions. There are more recent classifications like Kurka's one which is based on topological properties.[1] In this paper, we will be interested by Rapaport's algebraic classification stemming from two syntactical notions: from the dynamical system nature of CA, Rapaport introduced the sub-automaton notion, which is an intuitive but restrictive notion of simulation, together with the grouping operation so that his classification is invariant by scale changing, which seems very natural.[6] This classification is well formalized, but there are few results on it. On the bottom of this classification, there are very natural algorithmic classes, like nilpotent or periodical CA, but few things about the higher classes or the position of rules like 54.

In the first section, we will recall Boccara's results about the rule 54. The second one will give the main results of this paper: the group definition, its different interpretations and the consequences on particles interaction. In the third section, we will define Rapaport's definition and recall some results about this classification, then in the fourth section, we will analyze what the fact to be at an infinite level in it means. In the last section, we will see how to deduce the position of a 3 state CA above an infinite number of classes in Rapaport's classification from the previous results.

1 The Particle Like Behavior of the Rule 54

Definition 1.1 *A* **unidimensional cellular automaton** *is a couple* (Q, δ) *where* Q *is a finite set of states and* $\delta : Q^3 \longrightarrow Q$ *is a transition function. A configuration* $\Gamma \in Q^{\mathbb{Z}}$ *of* (Q, δ) *is a function from* $\mathbb{Z}$ *into* Q *and its global transition function* $G_\delta : Q^{\mathbb{Z}} \longrightarrow Q^{\mathbb{Z}}$ *is such that* $(G_\delta(\Gamma))(i) = \delta(\Gamma(i-1), \Gamma(i), \Gamma(i+1))$.

For two states (0 and 1) cellular automata, we will use Wolfram's notation: they are represented by an integer between 0 and 255 such that the transition function of the CA number i whose writing in base 2 is $i = \overline{a_7 a_6 a_5 a_4 a_3 a_2 a_1 a_0}^2$ satisfies:

$$\begin{array}{ll} \delta_i(0,0,0) = a_0 & \delta_i(1,0,0) = a_4 \\ \delta_i(0,0,1) = a_1 & \delta_i(1,0,1) = a_5 \\ \delta_i(0,1,0) = a_2 & \delta_i(1,1,0) = a_6 \\ \delta_i(0,1,1) = a_3 & \delta_i(1,1,1) = a_7 \end{array}$$

Let us remark that CA with different numbers may have the same behavior by switching the states 0 and 1, for instance $184 = \overline{10111000}^2$ and $226 = \overline{11100010}^2$.[3] We will speak about the cellular automata $54 = (\{0,1\}, \delta_{54})$ or equivalently of the rule 54.

The binary writing of 54 is ($54 = \overline{00110110}^2$), so its transition function is the following:

$$\delta_{54}(x,y,z) = \begin{cases} 1 & \text{if}(x,y,z) = \begin{cases} & (0,0,1) \\ \text{or} & (1,0,0) \\ \text{or} & (0,1,0) \\ \text{or} & (1,0,1) \end{cases} \\ 0 & \text{else} \end{cases}$$

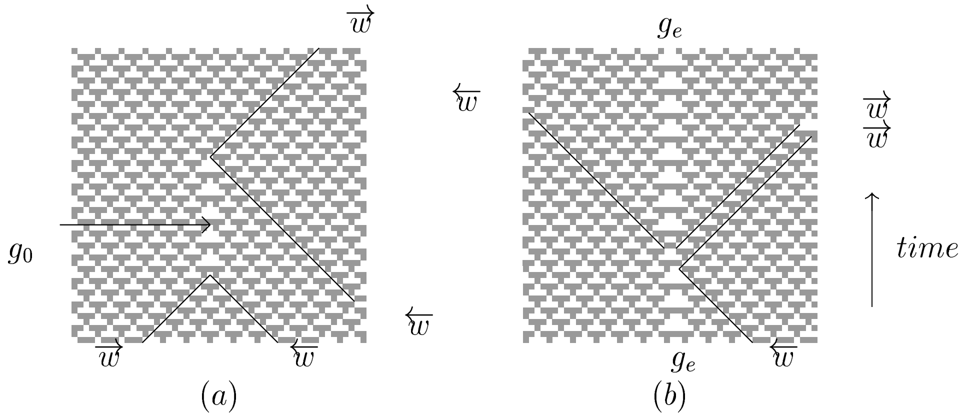


Figure 1: The collisions between a w particle and a g_0 or g_e particle.

The rule 54 is well known thanks to Boccara's article.[2] This article propose an interpretation of 54 space-time diagrams as particle interaction that we will recall here. By observing a space-time diagram of the rule 54 on a random configuration, we always see a kind of background with space and time period 4 (see figures 1 and 2). The background apparition is usually very fast (less than ten iterations are enough) and still mysterious, we have no explanation of this phenomenon. We can also distinguish three kind of particles. Two of them do not move and have a time period equal to 4. According to Boccara, we will call them g_0 (see figure 1 (a)) and g_e (see figures 1 (b) and 2). There is also a particle w which moves toward the right or toward the left at maximal speed 1. We will denote $\overleftarrow{w}$ the w particles going toward the left and $\overrightarrow{w}$ those going toward the right.

Interactions between these particles are rather rich, for instance, as we can see on the figure 1 (a), we have:

$$\begin{array}{ccccc} \overrightarrow{w} & + & \overleftarrow{w} & \longrightarrow & g_0 \\ \overrightarrow{w} & + & g_0 & \longrightarrow & \overleftarrow{w} \\ \overleftarrow{w} & + & g_0 & \longrightarrow & \overrightarrow{w} \end{array}$$

The figure 1 (b) represents the collision of a particle w with a particle g_e. On the diagrams (c) and (d) of the figure 2, two g_e particles close together are annihilated.

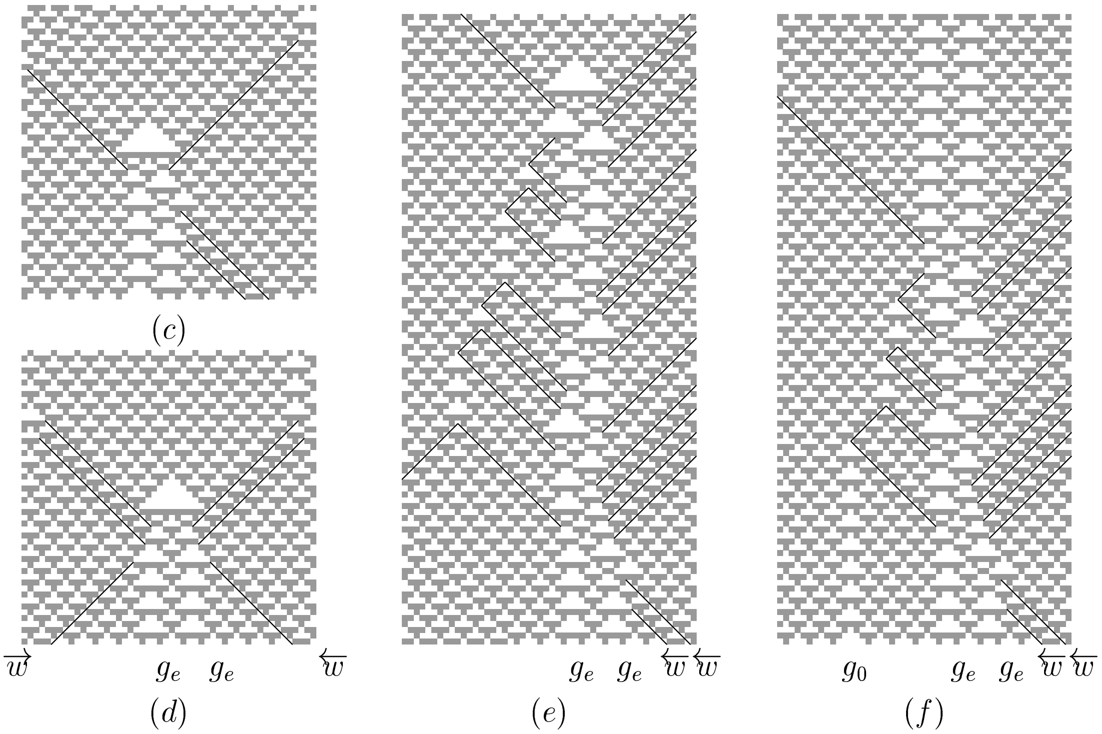


Figure 2: Annihilation of a couple of g_e particles, the radiating particle.

Two close particles g_e interacting with a particle w can produce another fixed particle (diagram (e)) with time period 32 which emits continuously w particles (four $\overrightarrow{w}$ and four $\overleftarrow{w}$ during each cycle). This particle can by destroyed by a particle w, which is delayed by the radiated particles, but collides it and either annihilates it (as on diagram (e)), or creates two g_e particles (as on diagram (f)).

2 The Rule 54

For Boccara, there is only one background on a space-time diagram of 54 on a random configuration because the areas containing no particle are filled by the same pattern. We can obtain many other backgrounds, that is other patterns, on 54 space-time diagrams with specific initial conditions, but experimentally, these other backgrounds never appear from a random configuration.

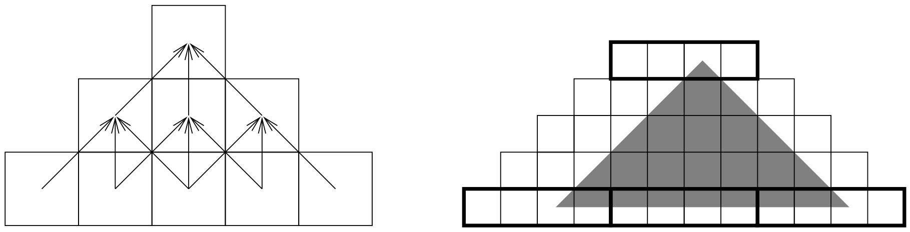
We will now use a more precise notion of background and say that two areas are filled by the same background if there exists a tiling of the whole plane $\mathbb{Z}^2$ which coincide on both areas with the space-time diagram. To avoid confusion we will always speak of Boccara's background to denote the fisrt notion and simply of a background for our new notion. Actually, many backgrounds (for this new definition) may have the same pattern (with an offset) and are also considered as only one Boccara's background.

The background, in Boccara's sense, that appears on space-time diagrams with a random initial configuration, splits in eight backgrounds, that we will point out by grouping the rule 54 4 by 4 (we will define the grouping operation just below), before explaining how to recognize them directly on 54 space-time diagrams.

2.1 Grouped automata

Being given a cellular automaton (Q, δ), and a cell on its space-time diagram, its dependence graph is a cone (cf figure 3 (a)). We can thus recursively define the function $\delta^n : Q^{2n+1} \longrightarrow Q$ which computes the central cell state of a block of $2n+1$ cells after n steps of computation:

$$\begin{aligned} \delta^1(x_{-1}, x_0, x_1) &= \delta(x_{-1}, x_0, x_1) \\ \delta^{n+1}(x_{-(n+1)}, ..., x_{(n+1)}) &= \delta^n(\delta(x_{-(n+1)}, x_{-n}, x_{-n+1}), ..., \delta(x_{n-1}, x_n, x_{n+1})) \end{aligned}$$



(a) (b)

Figure 3: (a) The dependence relation between the states in the space-time diagram of a CA.
(b) A representation of the grouping operation (four by four) of a cellular automaton.

By grouping the states k by k and by making k steps of computation of the initial CA in only one step (figure 3 (b)), the k central cells k steps later depend only of the $3k$ known cells at the preceding step), we can define a new cellular automaton with (exponentially according to k) more states. This operation does not modify the behavior of the automaton. On the space-time diagrams, this transformation looks like a change of scale.

Definition 2.1 *Let $\mathcal{A}$ be a cellular automaton, the automaton grouped n times of $\mathcal{A}$ is the automaton denoted $\mathcal{A}^n = (Q_\mathcal{A}, \delta_\mathcal{A})^n = (Q^n_\mathcal{A}, \delta_{\mathcal{A}^n})$ such as $\forall i \in \{1..., N\}$ $(\delta_{\mathcal{A}^n}(\vec{x}, \vec{y}, \vec{z}))_i = {\delta_\mathcal{A}}^n(x_i..., x_n, y_1..., y_n, z_1..., z_i)$ with the following conventions: an element $\vec{q} \in Q^n_\mathcal{A}$ denotes $(q_1..., q_n)$ and $(\vec{q})_i = q_i$ represents $\vec{q}$ i^{th} coordinate.*

Notation 2.1 *We will use the same notation $\mathcal{A}^i$ for the automata with two states, thus 54^4 will represent the automaton 54 grouped 4 times and not the number.*

2.2 The 8 backgrounds of the rule 54

Boccara's interpretation of the rule 54 introduces a background in Boccara's sense with space and time period 4. So in the cellular automaton 54^4 (i.e. the rule 54 grouped four times), this background appears uniform (i.e. composed by only one grouped state). Depending on the chosen origin, we get 8 stable backgrounds of 54^4 that are also backgrounds of 54:

$$\begin{array}{ll} b_{0,0} = (0,0,0,1) & b_{0,1} = (1,0,1,1) \\ b_{1,0} = (0,0,1,0) & b_{1,1} = (0,1,1,1) \\ b_{2,0} = (0,1,0,0) & b_{2,1} = (1,1,1,0) \\ b_{3,0} = (1,0,0,0) & b_{3,1} = (1,1,0,1) \end{array}$$

For some reasons that we will explain below, we indexed the backgrounds by the elements of the group G of presentation $< a, b \mid ab = ba, a^4 = 1, b^2 = a^2 >$[1] (which has 8 elements). For simplicity, we denote $b_{k,l}$ the background associated to the element $a^k b^l$ of G. For instance, if $k = 4$ and $l = 2$, $b_{4,2} = b_{0,2} = b_{2,0}$. We will in the following adopt this last representation (when the first index is between 0 and 3 and the second one between 0 and 1).

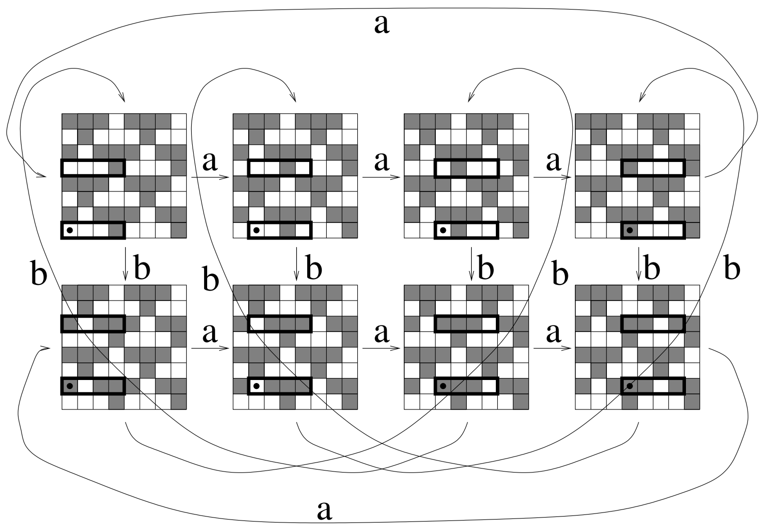


Figure 4: The group indexing the 8 backgrounds of 54^4.
a generator corresponds to a shift toward the right of the grouping origin (indicated by a point) and b to a shift toward the top.

Let us now explain why this group G appears. Let us consider Boccara's background of the rule 54 on random configuration and let us group it 4 by 4 to obtain the uniform background composed by $b_{0,0} = (0,0,0,1)$. By doing this, we implicitly chose an origin (see figure 4). If we shift the origin of one cell toward the right, we now obtain a uniform background composed by $b_{1,0} = (0,0,1,0)$. We will represent this shift toward the right by the a generator of G. Some new shifts toward the right give respectively

[1]This group is finite and Abelian, it can also be written as a finite product of cyclic groups, here $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ by considering the generators a and ab.

$b_{2,0} = (0,1,0,0)$, $b_{3,0} = (1,0,0,0)$ then again $b_{4,0} = b_{0,0} = (0,0,0,1)$, which is logical because Boccara's background has spatial period 4 and this justifies the relation $a^4 = 1$ because after four shifts toward the right we come back to the same background. The b generator of G represents a shift of the origin toward the top. Obviously, a shift toward the top then toward the right is equivalent to a shift toward the right and then toward the top, so we can deduce the commutativity of a and b: $ab = ba$. A first shift toward the top from $b_{0,0} = (0,0,0,1)$ leads us to a new background: $b_{0,1} = (1,0,1,1)$. After a second shift toward the top, we find a known background $b_{0,2} = b_{2,0} = (0,1,0,0)$, so we have the last relation $b^2 = a^2$. So the group G is built from the two operations (shifting the origin toward the right and toward the top), that allows to move from one background of 54^4 to another one.

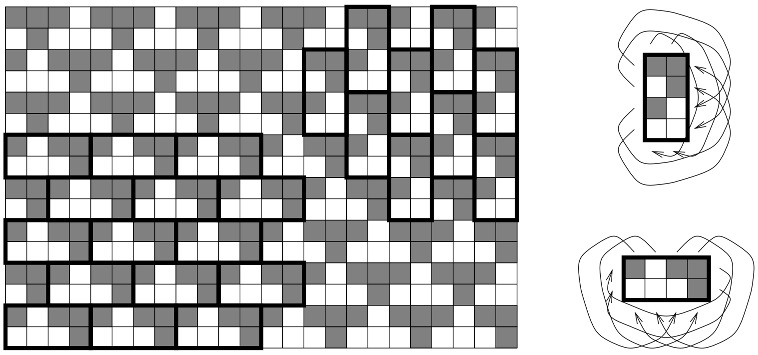


Figure 5: Two tilings of the 54 background with tiles of minimal size (8 cells). In both cases, the group associated to the tiling (the quotient group of $\mathbb{Z}^2$ which identifies the tiles) is the group G of presentation $< a, b \mid ab = ba, a^4 = 1, b^2 = a^2 >$.

In the previous explanation, the apparition of the group G may seem a bit magic. We will here justify the existence of the 8 backgrounds of 54 and link them to the group G with a geometrical point of view. Actually, we will associate to each Boccara's background, a group. The existence of this group is guaranteed by what we implicitly mean by Boccara's background, some kind of periodicity. So for each Boccara's background we can define T, the translation group of $\mathbb{Z}^2$ letting the pattern invariant. We can also associate to each Boccara's background the quotient of the Abelian group $\mathbb{Z}^2$ by the group T and this group is characteristic of the pattern drawn on the space-time diagram of Boccara's background.

If we now consider the Boccara's background that appears on space-time diagrams with random initial configuration, the group T is generated by the horizontal translation of 4 cells and the diagonal translation of two cells toward the bottom and two cells toward the right. In addition, by choosing the notation $b_{k,l}$ with $0 \leq k \leq 3$ and $0 \leq l \leq 1$ for the 8 backgrounds of 54^4, we implicitly describe a tiling of $\mathbb{Z}^2$ with minimal size tiles and respecting the pattern. More precisely, we chose step by step one representant for each equivalent class of the quotient of $\mathbb{Z}^2$ by the group T, that led us to the tiling represented in the bottom left corner of the figure 5. The choice of this presentation of the group G is arbitrary, we could as well have chosen any other presentations, that is, geometrically, any other tilings with minimal size tiles. One of them is represented in the upper right corner of the same figure. In both cases, we can check that the group naturally associated to the tiling is the group G.

Moreover, the 8 backgrounds are indexed by the elements of the group G, thus this group naturally acts on the set of this 8 backgrounds. Actually, the most important to understand the rule 54 is not to know for each area which is its background but rather by which element of G we need to multiply the index of one of them to get the index of another one: the behavior of 54 with an initial configuration made up of $b_{0,0}$ and then of $b_{1,1}$ will be overall the same one as the evolution from a configuration made up of $b_{k,l}$ and then of $b_{k+1,l+1}$. Indeed, that corresponds to a shift, in the diagram of the rule 54 of the choice of the origin of the grouping operation of k cells toward the right and of l cells toward the bottom.

To illustrate this, we will use a filter on 54 space-time diagrams. In order to easily recognize the 8 backgrounds, we will color the diagrams with 8 colors, one color for each background. The idea is the following, the color of one cell will not only depend on its state, but also on its two neighbor states and on its position modulo 4 in space and time. Let $(s, t) \in \mathbb{Z} \times \mathbb{N}$ be the coordinates of a cell in the space-time diagram, we color this cell with one of the eight colors $c_{k,l}, 0 \leq k \leq 3, 0 \leq l \leq 1$ in the following array:

100 $c_{0-s-t[4],0-t[2]}$	010 $c_{1-s-t[4],0-t[2]}$	001 $c_{2-s-t[2],0-t[2]}$	000 $c_{3-s-t[4],0-t[2]}$
101 $c_{0-s-t[4],1-t[2]}$	110 $c_{1-s-t[4],1-t[2]}$	111 $c_{2-s-t[4],1-t[2]}$	011 $c_{3-s-t[4],1-t[2]}$

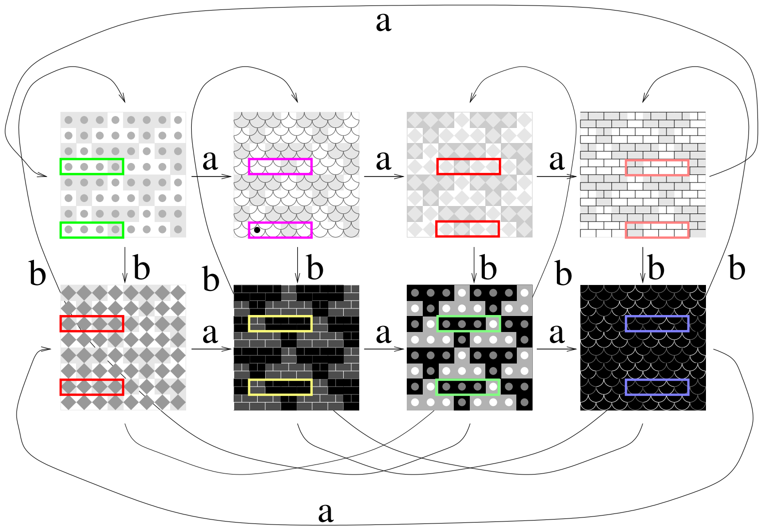


Figure 6: The natural action of the group G on the 8 colors.

It is easy to check that an area containing the background $b_{k,l}$ is filled by the color $c_{k,l}$ in the filtered configuration, except on the border of the area. So, the group G which acts on the set of backgrounds, also naturally acts on the set of colors. The colors we chose and the action of G on them are represented on figure 6. On a random configuration, a filtered space-time diagram looks like figure 7. We can see that particles

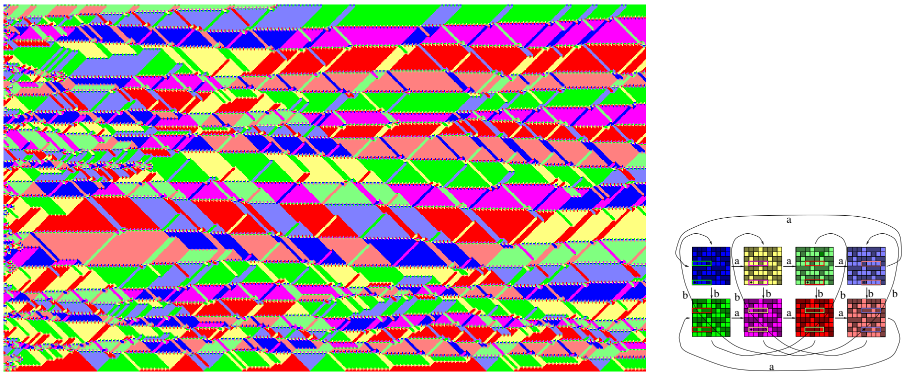


Figure 7: A filtered evolution on a random configuration, times go toward the right.

(w, g_0, g_e) do not evolve in an homogeneous background, but are in fact frontiers between different backgrounds. We could take this as a definition, and say that simple particles are frontiers between backgrounds.

We can now illustrate the previous remark and explain how the group action on the backgrounds (and so on the colors) will allow us to understand the particle interactions.

On the figure 8, we see that the particle w is in fact the frontier between two distinct backgrounds. Moreover, we get the index of the right background by multiplying the left background index by ab for a $\overrightarrow{w}$ particle and by $a^{-1}b$ for a $\overleftarrow{w}$ particle. As we explained before, this is independent of the left background because it corresponds to a translation of the space-time diagram. We can thus associate the element ab to the particle $\overrightarrow{w}$ and $a^{-1}b$ to the particle $\overleftarrow{w}$. We can also observe that when we have two parallel w particles, the leftmost background is the same as the rightmost one. We could have guess this by remarking that the order of ab and of $a^{-1}b$ is 2 in G (i.e. $(ab)^2 = (a^{-1}b)^2 = 1$).

On the figure 9, we see that we have to associate the element a^2 of G to the particle g_0. Again $(a^2)^2 = 1$ so when we have two parallel g_0 particles, the leftmost background is the same as the rightmost one. What about the collision of a g_0 with a $\overleftarrow{w}$ particle? At the impact point, we have to multiply the leftmost background index by a^2 and then ab and so by $a^3b = a^{-1}b$ to get the rightmost background index, the collision creates not surprisingly a $\overrightarrow{w}$ particle on the bottom left diagram. The elements associated to the particles w and g_0 generate a strict sub-group $\{1, ab, a^2, a^{-1}b\}$ of G. In fact, the behavior of a configuration containing only w and g_0 particles is completely understood, knowing the interaction of w with g_0. Thus, the main reason of 54 complexity is the last particle: g_e.

On the figure 10, we consider the g_e particle and associate it to the element a of G. We can see its interaction with a w particle and the strange radiating particle (that we observed on the figure 2 (e)). Of course, we cannot associate any group element to this "particle" because it does not separate two fixed backgrounds, our method does not give us any information about this behavior. But, we can now explain why it is impossible to annihilate only one g_e particle. Suppose that on the initial configuration, we have only one g_e particle with w and g_0 particles, we have to multiply the leftmost background index by a, a^{-1}, a^2b or b to get the rightmost one, if it was possible to annihilate the g_e particle, after a while, we would have only w and g_e particles ... so

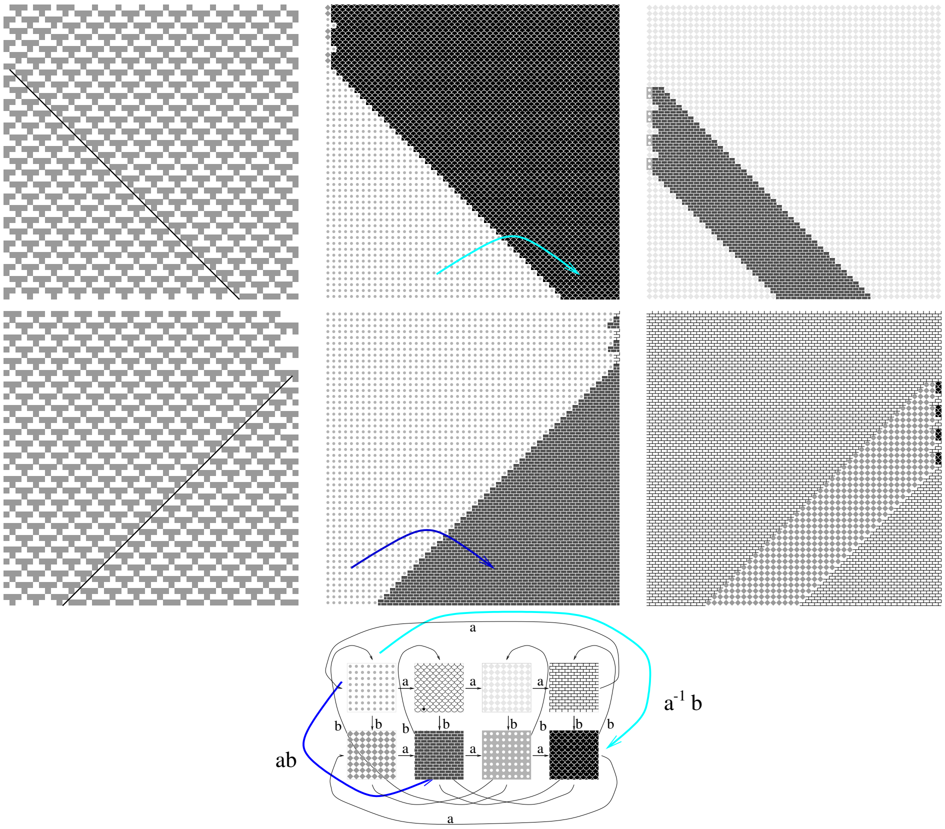


Figure 8: The particles $\overrightarrow{w}$ is linked to ab and $\overleftarrow{w}$ to $a^{-1}b$.

the leftmost background index should be multiplied only by 1, a^2, ab or $a^{-1}b$, that is impossible. Anyway, it is possible to annihilate two (more generally an even number) of g_e particles as we can see on the right diagrams (figure 10). In fact, the group G gives us a kind of conservation law on the particle interactions, which allow to prove negative results on them.

We associated four group elements to the four known particles, this raises a natural question: what about the 3 other elements which are distinct of the unit, and indirectly are there other particles? The answer to the first question is partially given by the figure 11. As there is no configuration with one background of index α on the left and then immediately the background of index αb or $\alpha a^2 b$, we have represented the two possibilities with less than 2 transition cells. Not surprisingly, as $b = a(a^{-1}b) = a^{-1}(ab)$ and $a^2b = a(ab) = a^{-1}(a^{-1}b)$, we observe the same behavior that after a collision between a g_e and a w particle (represented on the bottom left diagram). We can see on the figure 11 that for the 3 other elements, if we just study the fastest transitions between the backgrounds, none of the possibilities is stable, i.e. it always generates several known particles. Unfortunately, nothing proves that there is not a (possibly very huge) stable particle between two of these backgrounds. As for the background generation, we cannot prove anything, we can just observe that such a particle does not appear on random configurations.

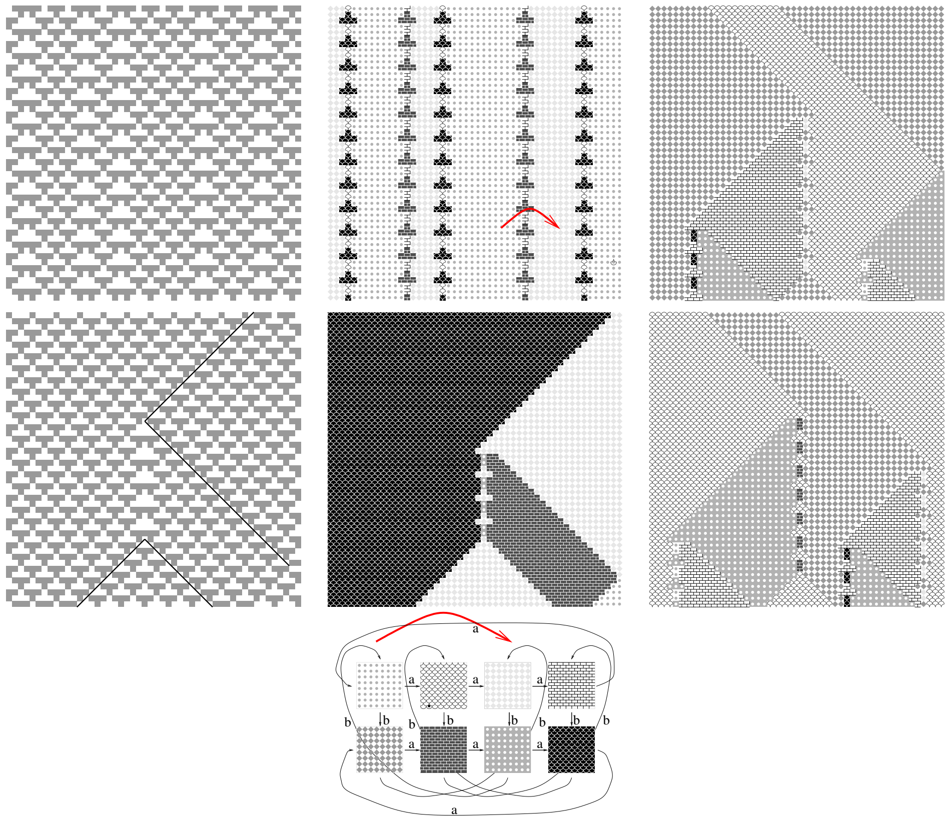


Figure 9: The particle g_0 is linked to a^2.

The first application of the group introduction was the proof of the impossibility to annihilate only one g_e, we will now use this method to prove that cellular automata with few states may be "complicated" in Rapaport's classification, i.e. above an infinite number of classes. Let us first define Rapaport's classification and recall some results about it.

3 Definitions

3.1 Rapaport's algebraic classification

Inspired by the concept of subsystem in dynamic systems theory, the concept of sub-automaton is naturally introduced for cellular automata. Distinct colors being assigned to each state of $\mathcal{A}$, $\mathcal{A}$ is a sub-automaton of $\mathcal{B}$ if it is possible to assign distinct colors (including the colors assigned to the states of $\mathcal{A}$) to each state of $\mathcal{B}$ so that all the space-time diagrams of $\mathcal{A}$ are space-time diagrams of $\mathcal{B}$. Formally, that results in the existence of an injection from the states of $\mathcal{A}$ into those of $\mathcal{B}$ which is compatible with the transition functions operating on the states of $\mathcal{A}$ and $\mathcal{B}$:

Definition 3.1 Sub-automaton
Being given $\mathcal{A} = (Q_\mathcal{A}, \delta_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B}, \delta_\mathcal{B})$ two cellular automata, $\mathcal{A}$ is a sub-

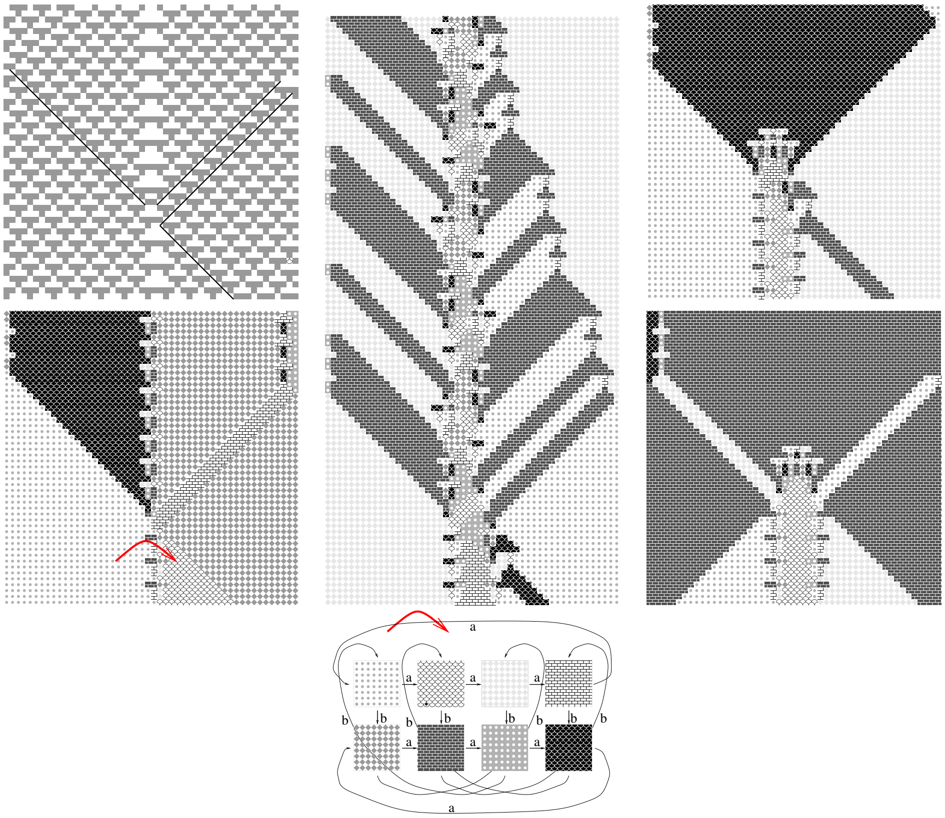


Figure 10: The particles g_e is linked to a.

automaton of $\mathcal{B}$ ($\mathcal{A} \subseteq \mathcal{B}$), if and only if there is an injective function $\iota : Q_\mathcal{A} \longrightarrow Q_\mathcal{B}$ such as $\forall x, y, z \in\ Q_\mathcal{A}\ \delta_\mathcal{B}(\iota(x), \iota(y), \iota(z)) = \iota(\delta_\mathcal{A}(x, y, z))$.

We will admit the following obvious properties with respect to the grouping operation:

Property 3.1 *If $\mathcal{A} \subseteq \mathcal{B}$ then for all n, $\mathcal{A}^n \subseteq \mathcal{B}^n$.*

Property 3.2 *Let $\mathcal{A}$ be a CA, for all $n, m \in \mathbb{N}^*$ there exists a bijection between the states of $(\mathcal{A}^n)^m$ and those of $\mathcal{A}^{nm}$ compatible with their transition functions.*

To understand the space-time diagram of a cellular automaton, the human eye typically adapts the scale to reveal the principal phenomena without taking into account the non relevant transition behavior. From this observation, Rapaport connects the concept of sub-automaton to that of grouping to define the pre-order below.[6]

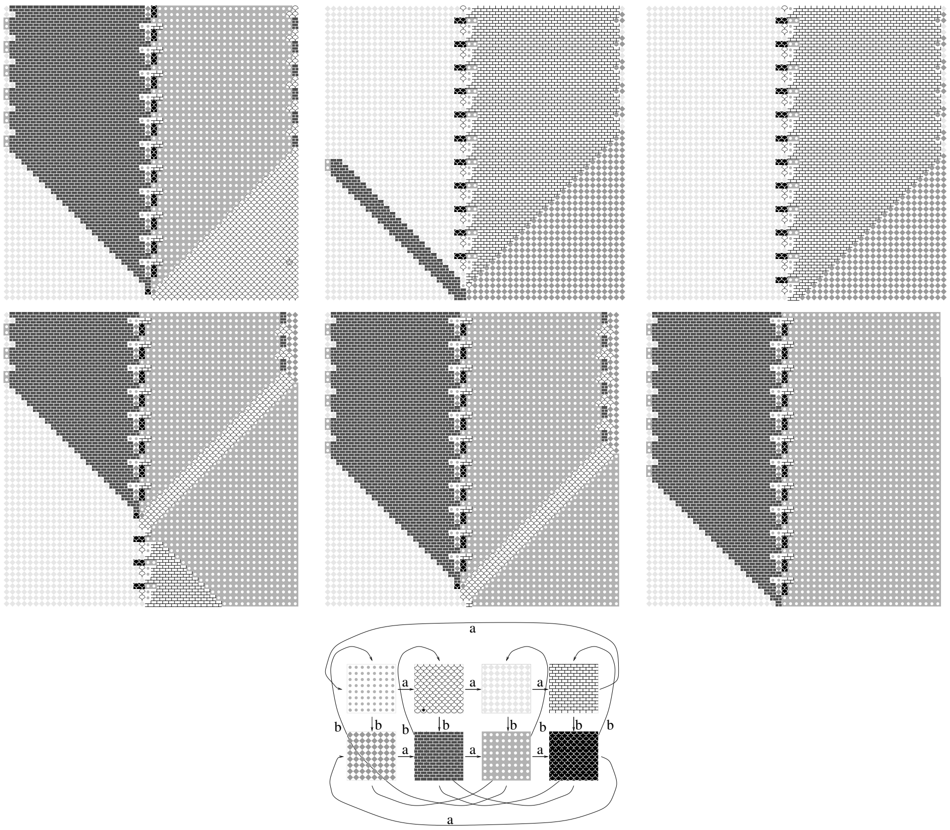


Figure 11: The other background interactions, none of them are stable.

Definition 3.2 Pre-order introduced by Rapaport
Being given $\mathcal{A}$ and $\mathcal{B}$ two cellular automata, $\mathcal{A} \leq \mathcal{B}$ if and only if there are integers i and j such as $\mathcal{A}^i \subseteq \mathcal{B}^j$, i.e. $\mathcal{A}^i$ is a sub-automaton of $\mathcal{B}^j$.

It is indeed a pre-order because the reflexivity is obvious, as for transitivity, it results from the admitted properties: if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{C}$ then there exists n_1, m_1, n_2, m_2 such as $\mathcal{A}^{n_1} \subseteq \mathcal{B}^{m_1}$ and $\mathcal{B}^{n_2} \subseteq \mathcal{C}^{m_2}$ thus $\mathcal{A}^{n_1 n_2} \subseteq \mathcal{C}^{m_1 m_2}$. This pre-order induces an equivalence relation $\sim$, and the set $(CA/\sim, \leq)$ is an order by construction.

Let us consider the CA $\mathcal{S} = (\{\_, l, r\}, \delta_{\mathcal{S}})$ such that l state goes toward the left, r state goes toward the right and when an l state meets an r one, they annihilate together. We can see on the figure 12 that $\mathcal{S}$ is less than the rule 184 because we can represent all its space-time diagrams as 184^2 (184 whose states are grouped 2 by 2) space-time diagrams. We can also see on the same figure that $\mathcal{S}$ is also less than the rule 54, but we have to group 54 four by four. Let us notice that $184 \not\leq 54$.

3.2 Some results about Rapaport's order

Let us first define one state and additive CA:

Definition 3.3

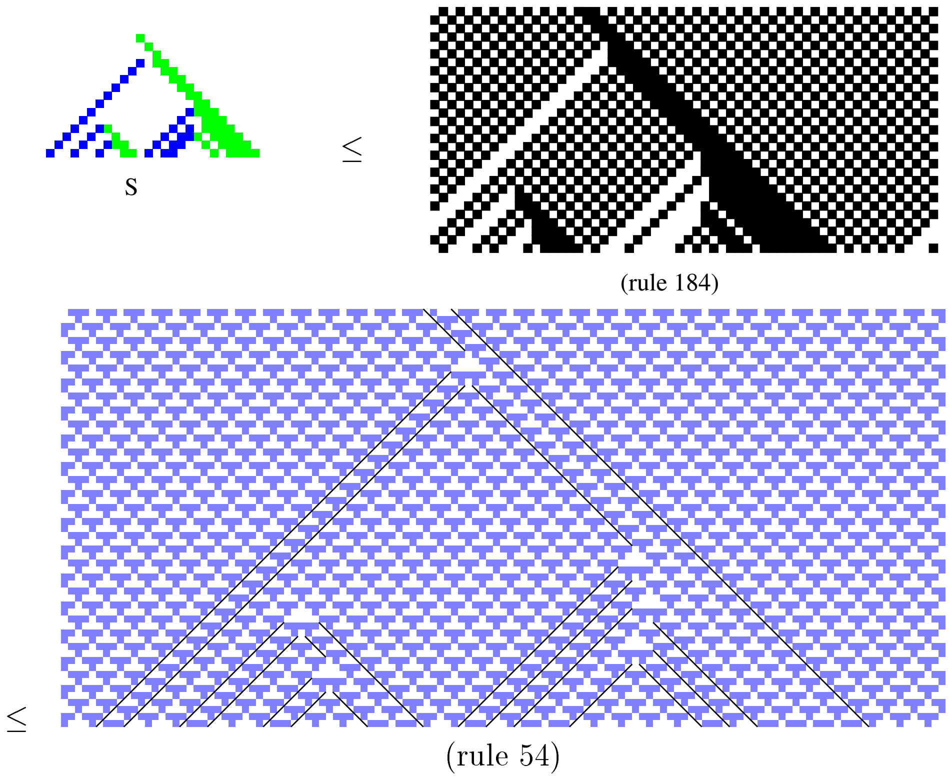


Figure 12: $\mathcal{S} \leq 184, 54$.

- *One state cellular automaton looks like* $\nVdash = (\{s\}, \delta_s)$ *with* $\delta_s(s, s, s) = s$. *Of course, all the cells are always in the state* s. *We will denote* $\nVdash$ *a CA with one state and* $\overline{\nVdash}$ *the class of all one state cellular automata;*
- *Among CA which are easy to study, we find the CA families amenable to algebraic analysis. Among them, we will mention the family of additive CA* $(\mathbb{Z}_p, \oplus)$ *such that* $\oplus(x, y, z) = x + y + z$ *in* $\mathbb{Z}_p$ *(we denote by* $\mathbb{Z}_n(= \mathbb{Z}/n\mathbb{Z})$ *the set of integers modulo* n *and conventionally, the notation* $\mathbb{Z}_p$ *will be exclusively used when* p *is prime).*

It has been proved by Rapaport, that there are very natural classes of CA on the bottom of his classification: the global minimum is the class $\overline{\nVdash}$ of one state CA, just above the minimal, we find the sets of nilpotent CA, of periodical CA, and of left (resp. right) shifts.[6] In addition, there is no maximum class and we can define two infinite chains with a common upper bound.[5] Furthermore, the additive CA $(\mathbb{Z}_p, \oplus)$ for p prime also belong to distinct classes just above the minimum of Rapaport's order (this result is obvious) and together with the next section results about the product operator, we can easily deduce again that there is no maximum. More interesting about these additive CA, they cannot be simulated in Rapaport's sense by CA with less than p states, thus, in some way, they are optimal in state number. Some of previous results are summarized in the diagram 13.

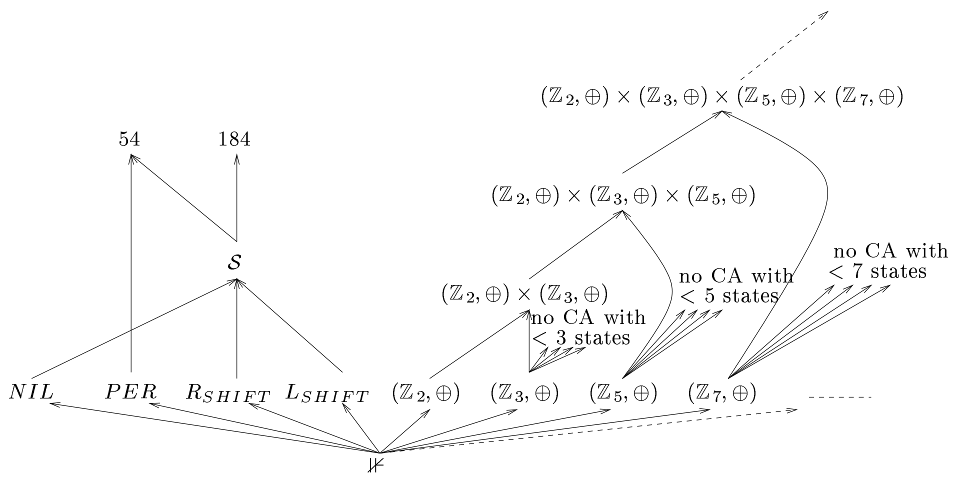


Figure 13: Some results about Rapaport's order.

3.3 Two composition operators on CA: product and union

3.3.1 Product

Let $\mathcal{A}$ and $\mathcal{B}$ be two CA, the product automaton is a cellular automaton which simulates in parallel one computation of $\mathcal{A}$ and one computation of $\mathcal{B}$.

Definition 3.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be two CA, their product automaton is $\mathcal{A}\times\mathcal{B} = (Q_{\mathcal{A}}\times Q_{\mathcal{B}}, \delta_{\mathcal{A}\times\mathcal{B}}))$ where $\delta_{\mathcal{A}\times\mathcal{B}}((x_{\mathcal{A}},x_{\mathcal{B}}),(y_{\mathcal{A}},y_{\mathcal{B}}),(z_{\mathcal{A}},z_{\mathcal{B}})) = (\delta_{\mathcal{A}}(x_{\mathcal{A}},y_{\mathcal{A}},z_{\mathcal{A}}),\delta_{\mathcal{B}}(x_{\mathcal{B}},y_{\mathcal{B}},z_{\mathcal{B}}))$.*

Considering the canonical injections, we obviously have the following property:

Property 3.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two CA, we have $\mathcal{A},\mathcal{B}\leq\mathcal{A}\times\mathcal{B}$.*

3.3.2 Union

As we can define an automaton whose states set is the product of two CA states set, it seems natural to define a cellular automaton whose states set is the union of two states sets. Unfortunately, the transition function choice for the product case is very natural, but this choice is no more obvious in the union case. So we will define a set of union automata rather than one union automaton:

Definition 3.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be two CA (with distinct states sets: $Q_{\mathcal{A}}\cap Q_{\mathcal{B}}=\emptyset$), the union set $\mathcal{A}\cup\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is composed by the CA $\mathcal{C}=(Q_{\mathcal{A}}\cup Q_{\mathcal{B}},\delta_{\mathcal{C}})$ whose transition function satisfies:*

$$\begin{cases} \textit{if} & x,y,z\in Q_{\mathcal{A}} & \textit{then} & \delta_{\mathcal{C}}(x,y,z)=\delta_{\mathcal{A}}(x,y,z)\\ \textit{if} & x,y,z\in Q_{\mathcal{B}} & \textit{then} & \delta_{\mathcal{C}}(x,y,z)=\delta_{\mathcal{B}}(x,y,z)\\ \textit{if} & y\in Q_{\mathcal{A}} & \textit{then} & \delta_{\mathcal{C}}(x,y,z)\in Q_{\mathcal{A}}\\ \textit{if} & y\in Q_{\mathcal{B}} & \textit{then} & \delta_{\mathcal{C}}(x,y,z)\in Q_{\mathcal{B}} \end{cases}$$

Let us notice that the CA which belong to $\mathcal{A} \cup \mathcal{B}$ are not necessary in the same class in the Rapaport's order. Anyway, all of them are higher than $\mathcal{A}$ and $\mathcal{B}$:

Property 3.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be two CA, for all $\mathcal{C} \in \mathcal{A} \cup \mathcal{B}$, we have $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ and thus $\mathcal{A}, \mathcal{B} \leq \mathcal{C}$.*

4 Application to Rapaport's Classification

4.1 Complicated CA for $(CA/\sim, \leq)$ order

We have many results about the "simple" classes for Rapaport's order, i.e. the classes on the bottom of the classification. We will now try to understand "complicated" classes for this order, that is classes above an infinite number of different classes. As the CA in simple classes are algorithmically simple, Rapaport conjectures that CA in complicated ones must be algorithmically complicated. Unfortunately, this is not really linked to algorithmic complexity, but rather to self-similarity: nilpotent, periodic and shift behaviors are both algorithmically simple and "self-similar" (that we will define formally below), and we will show that non self-similar behavior leads to complexity in Rapaport's order. Actually, we will prove that each CA which has a non self-similar sub-automaton is above an infinite number of different classes. The problem is that the rule 54 is not self-similar but it seems that all its sub-automata are self-similar. We will also prove that if we add an additional state which does nothing interesting (thanks to the union with a single state CA) to a non self-similar CA, the resulting CA is above an infinite number of different classes.

In fact, in the order $(CA/\sim, \leq)$, we will prove that if $\mathcal{A}$ is not self-similar, then there are an infinite number of different classes above each element of $\mathcal{A} \cup \mathbb{1}$ (i.e. each class of an element of $\mathcal{A} \cup \mathbb{1}$ is above an infinite number of classes). Being non self-similar means having a different behavior at each scale, formally that means that all its grouped instances are in different classes in the classification:

Definition 4.1 *A CA $\mathcal{A}$ is not self-similar iff for all $i \neq j$, $\mathcal{A}^i \nsubseteq \mathcal{A}^j$.*

Remark 4.1 *This definition is natural because if we consider a CA which does not satisfy it, then there exist i and j such that $\mathcal{A}^i \subseteq \mathcal{A}^j$, so intuitively the set of all its space-time diagrams is stable by homothety of constant j/i. Of course, all the CA we know which are on the bottom of Rapaport's classification are self-similar, some of them have very simple space-time diagrams ($\mathbb{1}$, periodical CA, shifts) and the space-time diagrams of the other CA (additive CA) are composed by a superposition of fractals.*

Let us give an example of such a cellular automaton using the non self-similarity of the parabola. We will define the CA $\mathcal{P}$ which draw a parabola on its space-time diagram thanks to particles l (going toward the left) and r (going toward the right) between a fixed state $|$ and a state p shifted at each collision.

Definition 4.2 *Let $\mathcal{P} = (\{|, l, r, P, p, \_\}, \delta_{\mathcal{P}})$ be the CA whose transition function is given by:*

$$
\begin{array}{rlrl}
\delta_{\mathcal{P}}(\_,\_,\_) = & \_ & \delta_{\mathcal{P}}(\_,\_,l) = & l \\
\delta_{\mathcal{P}}(\_,|,\_) = & | & \delta_{\mathcal{P}}(|,\_,l) = & l \\
\delta_{\mathcal{P}}(\_,|,l) = & | & \delta_{\mathcal{P}}(|,l,\_) = & r \\
\delta_{\mathcal{P}}(r,\_,\_) = & r & \delta_{\mathcal{P}}(\_,p,\_) = & p \\
\delta_{\mathcal{P}}(r,\_,p) = & r & \delta_{\mathcal{P}}(\_,P,\_) = & l \\
\delta_{\mathcal{P}}(r,p,\_) = & P & \delta_{\mathcal{P}}(P,\_,\_) = & p \\
\delta_{\mathcal{P}}(*,*,*) = & \_ & \multicolumn{2}{c}{\textit{in all the other cases}}
\end{array}
$$

This is an example of initial configuration which draws a parabola:

$$
\begin{array}{cccc|ccccccccc}
\cdots & \_ & \_ & \_ & \_ & \_ & \_ & l & \_ & p & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & \_ & \_ & l & p & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & \_ & \_ & P & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & \_ & r & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & r & \_ & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & r & \_ & \_ & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & r & \_ & \_ & \_ & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & l & \_ & \_ & \_ & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & l & \_ & \_ & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & l & \_ & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & \_ & l & p & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & \_ & P & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & r & p & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & r & \_ & p & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & r & \_ & \_ & p & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & l & \_ & \_ & p & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & l & \_ & p & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & l & p & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & \_ & P & \_ & \_ & \_ & \_ & \_ & \cdots \\
\cdots & \_ & \_ & \_ & \_ & r & p & \_ & \_ & \_ & \_ & \_ & \cdots
\end{array}
$$

To prove the non self-similarity of $\mathcal{P}$ we first have to check that there is no other algorithm to draw a parabola (it is easy since all the other state interactions lead to annihilation) and then remark that this CA can only draw a parabola of fixed parameter to conclude by refutation: else the set of all space-time diagrams would be stable by some homothety, so its restriction to space-time diagrams containing a parabola would also be stable. Thus a discrete parabola of fixed parameter would be stable by some homothety, that is impossible since continuous parabola are not self-similar.

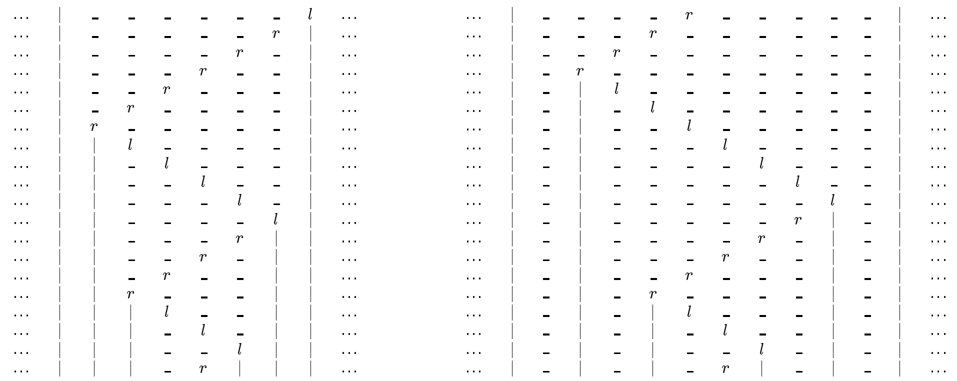
Let us notice that $\mathcal{P}$ is itself above an infinite number of classes since $\mathcal{P}^2$ has a non similar sub-automata.

The previous reasoning would not work if $\mathcal{P}$ was able to draw parabolas of different parameters, although this seems algorithmically more complicated to draw many kind of parabola instead of only one. Let us give such a CA $\mathcal{P}'$:

Definition 4.3 *Let $\mathcal{P}' = (\{|, g, d, \_\}, \delta_{\mathcal{P}'})$ be a CA drawing parabolas of different parameters. The parabolas are drawn by a signal (l toward the left and r toward the right) going and back between $|$ walls that are erased by the collision. Its transition function is the following:*

$$
\begin{array}{rlrl}
\delta_{\mathcal{P}'}(\_,\_,\_) = & \_ & \delta_{\mathcal{P}'}(\{\_,|,l\},|,\{\_,|,l\}) = & | \\
\delta_{\mathcal{P}'}(\{\_,|,l\},\_,l) = & l & \delta_{\mathcal{P}'}(r,\_,\{\_,|,r\}) = & r \\
\delta_{\mathcal{P}'}(\_,|,l) = & r & \delta_{\mathcal{P}'}(r,|,\_) = & l \\
\delta_{\mathcal{P}'}(|,|,l) = & r & \delta_{\mathcal{P}'}(r,|,|) = & l \\
\delta_{\mathcal{P}'}(*,*,*) = & \_ & \textit{in all the other cases} &
\end{array}
$$

These are two initial conditions drawing parabolas:



We can observe that this CA has the same behavior at each scale, that is for all i, $\mathcal{A} \subseteq \mathcal{A}^i$, we just have to consider the injection which insert $i-1$ "_" between $\vec{x}$ elements: $\vec{x} \longmapsto (x_1, \_, ..., \_, x_2, \_, ..., \_, ..., x_i, \_, ..., \_)$. Actually $\mathcal{P}'$ is at a finite high in Rapaport's classification. Thus be "complex" in Rapaport's classification seems more related to non self-similarity than to algorithmic complexity since drawing many kinds of parabolas seems algorithmically more complicated than drawing only one.

Let us now prove that a non self-similar CA with one more state well chosen is above an infinite number of different classes:

Theorem 4.1 *Let $\mathcal{A}$ be such that $i \neq j$, $\mathcal{A}^i \not\subseteq \mathcal{A}^j$ and without any fixed state among its grouped instance (i.e. there is no state $q \in Q_{\mathcal{A}^k}$ such that $\delta_{\mathcal{A}^k}(*, q, *) = q$). Then for all $\mathcal{C} \in \mathcal{A} \cup \nVdash$, there is an infinite set of distinct classes $\overline{\mathcal{D}}$ such that:*

$$\mathcal{A} < \mathcal{D} < \mathcal{C}$$

Proof 4.1 Let us consider the CA $\mathcal{D}_n = (Q_n, \delta_n)$ whose state set is $Q_n = Q_{\mathcal{A}} \times (Q_{\mathcal{A}} \cup Q_{\nVdash})^{n-1}$ and whose transition function $\delta_n = (\delta_{\mathcal{C}})^n_g\big|_{Q_n}$ is the restriction of $\delta_{\mathcal{C}^n}$ to Q_n. It is obvious that for all n, $\mathcal{A} \leq \mathcal{D}_n \leq \mathcal{C}$, we just have to consider the canonical injections to check that $\mathcal{A}^n \subseteq \mathcal{D}_n \subseteq \mathcal{C}^n$.

Let us now prove that $n < m$, $\mathcal{D}_m \not\leq \mathcal{C}_n$. By refutation, else, there exist i and j such that $\mathcal{D}^i_m \subseteq \mathcal{C}^j_n$. So, $\mathcal{A}^{mi} \subseteq \mathcal{D}^i_m \subseteq \mathcal{C}^j_n$. In addition, the injection ϕ from $Q^{mi}_{\mathcal{A}}$ into $(Q_{\mathcal{A}} \times (Q_{\mathcal{A}} \cup Q_{\nVdash})^{n-1})^j$ satisfies $\phi(Q^{mi}_{\mathcal{A}}) \subset Q^{nj}_{\mathcal{A}}$ because we assumed that no state of $\mathcal{A}^{mi}$ is fixed. Thus, $\mathcal{A}^{mi} \subseteq \mathcal{A}^{nj}$, and by hypothesis, $mi = nj$. But then, $\mathcal{D}^i_m \subseteq \mathcal{C}^j_n$

is impossible because $\mathcal{C}_m^i$ has $|Q_\mathcal{A}|^{mi} \times (|Q_\mathcal{A}|+1)^{(m-1)i}$ states, that is more than $\mathcal{D}_n^j$ which has $|Q_\mathcal{A}|^{nj} \times (|Q_\mathcal{A}|+1)^{(n-1)j}$ states (let us recall that $n < m$ and so $i < j$).

The CA $\mathcal{D}_n$ are thus all in different classes, there are also an infinite number of different classes between $\mathcal{A}$ and $\mathcal{C} \in \mathcal{A} \cup \not\vdash$. □

5 A 3 State CA (the Rule 54 Plus One State) at an Infinite Level

We will now apply the previous theorem to the rule 54 to prove there exists a 3 state CA at an infinite level in Rapaport's classification. To do that, we have to prove that the rule 54 has different behaviors at each scale.

5.1 The rule 54 non self-similarity

Let us show now that each grouped instance of the rule 54 is not injected into any other grouped instance of this rule. In the preceding section, in order to exhibit an automaton having this behavior, we chose the automaton $\mathcal{P}$ which draws a parabola. Indeed, the parabola is not self-similar, the behaviors of the grouped automata $\mathcal{P}^i$ are thus all different. The possibility of drawing parabolas with rule 54 is also a first indication of its "complexity". The diagram 14 (time goes toward the right) represents a first method to draw parabolas with rule 54: two w particles rebound between two g_e particles which are shifted at each collision.

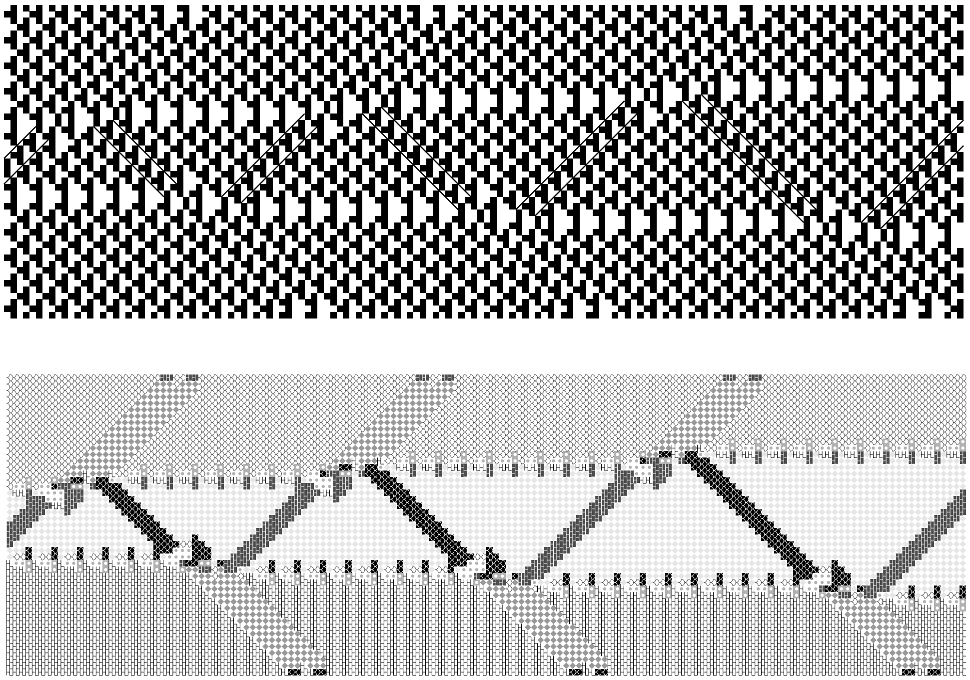


Figure 14: First algorithm drawing a parabola: two w particles between two g_e particles.

There are other algorithms to draw parabolas with the rule 54, for example that of $\mathcal{P}'$: only one w particle rebounds on particles g_0 which disappear after the collision (see figure 15).

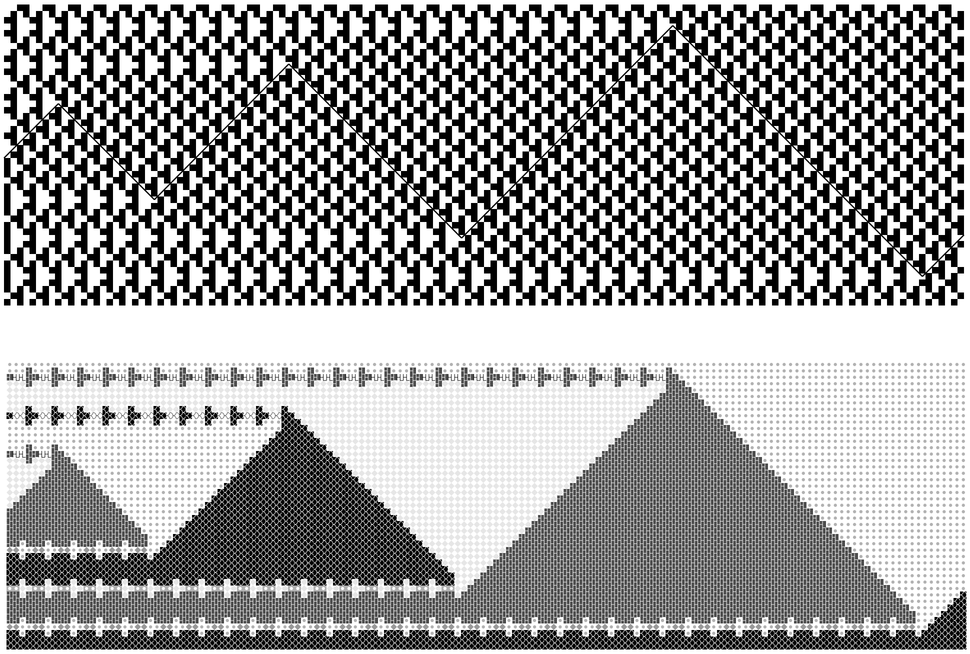


Figure 15: Second algorithm drawing a parabola: a particle w erases particles g_0.

If we move away the g_e particles, we obtain parabolas of different parameters. The argument used for the automaton $\mathcal{P}$ is thus not usable for the rule 54. Actually, a parabola of any parameter can be drawn on all the grouped automata 54^i. So, the proof is rather technical since the behavior of the automata 54^i cannot be analyzed as easily as that of $\mathcal{P}^n$ which had been especially built to check this property. The proof idea is thus based on the first algorithm (figure 14) which traces a parabola of fixed parameter. The idea is to show that the only possible injections of 54^i in 54^j $(i < j)$ simulate the particle $\overrightarrow{w}$, $\overleftarrow{w}$, g_0 and g_e by themselves (i.e. the image by the injection of a 54^i state, made of one particle surrounded by backgrounds, is a 54^j state, made of the same particle surrounded by backgrounds (possibly different)). For that, we will show that the backgrounds of the automaton grouped 4 times can only be sent on themselves and thus it is the same for particle w, g_0 and g_e. We will also show that for all i and j $(i < j)$, $54^{4i} \not\subseteq 54^{4j}$, which proves the expected result since $54^i \subseteq 54^j \Longrightarrow 54^{4i} \subseteq 54^{4j}$.

Lemma 5.1 *For all $i \neq j$, $54^{4i} \not\subseteq 54^{4j}$.*

Proof 5.1 The result is obvious if $i > j$ because 54^{4i} has more states than 54^{4j}. We now assume $i < j$.

Let us show that $54^{4i} \not\subseteq 54^{4j}$ by opposition and denote by ι the injection from $\{0,1\}^{4i} \longrightarrow \{0,1\}^{4j}$. Let us show first that there is an element $a^{k'}b^{l'}$ of G such as for all $a^k b^l \in G$,

$$\iota(\underbrace{b_{k,l}, b_{k,l} ..., b_{k,l}}_{i}) = (\underbrace{b_{k+k',l+l'}, b_{k+k',l+l'} ..., b_{k+k',l+l'}}_{j})$$

The first step is to show that a uniform background $(b_{k,l} ..., b_{k,l})$ of 54^{4i} is necessarily sent on a uniform background of 54^{4j}. The main characteristic of these backgrounds is their capacity to propagate some signal at speed 1 in both directions (i.e. to be crossed by the particle $\overrightarrow{w}$ and $\overleftarrow{w}$). A detailed study of 54 transition function shows that only the triplets 00? and 01? (resp. ?00 and ?10) depend on the third state (that 00? depends on the third state exactly means $\delta_{54}(0,0,0) \neq \delta_{54}(0,0,1)$, in other words $\delta_{54}(0,0,.)$ is bijective), it is also the only possibility to transmit a signal at speed 1 towards the left (resp. towards the right). Thus 54^{4j} space-time diagram with the uniform configuration (made up only of state $\iota(b_{k,l})$) as initial configuration must contain a diagonal towards the left made up of 00 or of 01 and one diagonal towards the right made up of 00 or of 10 to enable a particle crossing at speed 1. In fact, the diagonals are necessarily made up of only one sequence (00, 01 or 10): the sequences 01 and 10 impose that the following sequence of the diagonal are respectively 01 and 10 since $\delta_{54}(0,0,1) = \delta_{54}(1,0,1) = \delta_{54}(1,0,0) = 1$. Moreover, the diagonal made up of 00 can only be crossed by a diagonal also made up of 00, thus 2 cases has to be considered: two diagonals made up of 00 or one diagonal (left) made up of 01 and one diagonal (right) made up of 10. The figure 16 represents these two possibilities in the case $j = 3$.

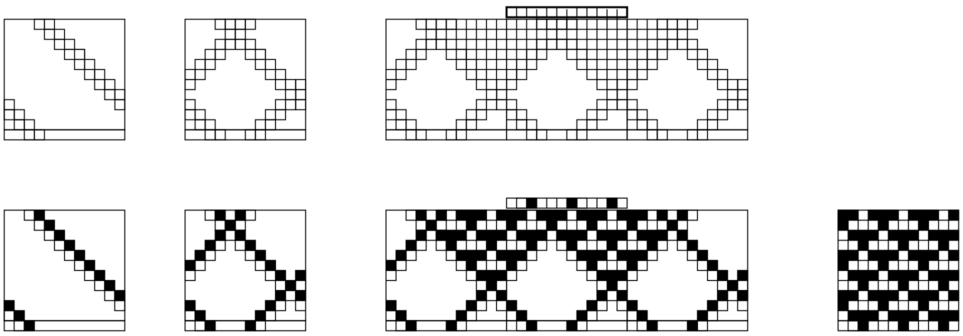


Figure 16: The stable states of 54^{12} allowing the transmission of signals at speed 1 in both directions:
From left to right, according to the composition of the left diagonal (00 or 01) the composition of the second diagonal is fixed (respectively 00 and 10). Knowing two diagonals, by stability, we can deduce the states, respectively $(0...,0)$ and $(b_{k,l}, b_{k,l}, b_{k,l})$ for some k and l depending on the diagonals positions, on our example: $(b_{1,0}, b_{1,0}, b_{1,0})$. On the rightmost diagram, we point out the complete pattern of the background $(b_{1,0}, b_{1,0}, b_{1,0})$.

As $\delta_{54^{4j}}(\iota(b_{k,l} ..., b_{k,l}), \iota(b_{k,l} ..., b_{k,l}), \iota(b_{k,l} ..., b_{k,l})) = \iota(b_{k,l} ..., b_{k,l})$, the presence of two diagonals allow to deduce $\iota(b_{k,l}..., b_{k,l})$, thus in the case of two diagonals of 00,

$$\iota(\underbrace{b_{k,l}..., b_{k,l}}_{i}) = (\underbrace{0, 0..., 0}_{4j})$$

and in the contrary case, we see that there exist k' and l' (according to the choice of the positions of the diagonals) such as

$$\iota(\underbrace{b_{k,l}..., b_{k,l}}_{i}) = (\underbrace{b_{k',l'}, b_{k',l'}..., b_{k',l'}}_{j})$$

It is clear that $\iota(b_{k,l}..., b_{k,l}) \neq (0, 0..., 0)$ because then, as ι is injective, $\iota(b_{k+2,l}..., b_{k+2,l}) = (b_{k',l'}..., b_{k',l'})$ for some k' and l' but

$$\delta_{54^{4i}}((b_{k,l}..., b_{k,l}), (b_{k,l}..., b_{k,l}), (b_{k+2,l}..., b_{k+2,l})) = (b_{k,l}..., b_{k,l})$$

$$\delta_{54^{4j}}((0..., 0), (0..., 0), (b_{k',l'}..., b_{k',l'})) = (b_{k',l'}..., b_{k',l'})$$

Lastly, taking into account specificities of the interactions between the various backgrounds let us show that if $\iota(b_{0,0}, ..., b_{0,0}) = (b_{k',l'}..., b_{k',l'})$, then $\iota(b_{k,l}..., b_{k,l}) = (b_{k+k',l+l'}..., b_{k+k',l+l'})$ (i.e. that ι preserves the group action on the backgrounds). It is easy to see that if $\iota(b_{0,0}..., b_{0,0}) = (b_{k',l'}..., b_{k',l'})$, then $\iota(b_{1,1}..., b_{1,1}) = (b_{k'+1,l'+1}..., b_{k'+1,l'+1})$ because $b_{k'+1,l'+1}$ is the only background whose interaction with $b_{k',l'}$ behaves like a shift towards the right-hand side (because of the creation of a particle $\overrightarrow{w}$, cf figure 8). By applying the same reasoning with other interaction, as we can now differentiate the particles g_0 and g_e thanks to the bombardment by some particle $\overrightarrow{w}$, an exhaustive study allows to show that the organization of 54^i backgrounds according to the group G is preserved by ι.

Thus, by juxtaposition of the backgrounds, we can create all the usual particles of the rule 54 on the automaton 54^i and we know that by ι they are simulated by particles in the same way on the automaton 54^{4j}. To conclude the proof, we notice that in the rule 54, the particle $\overleftarrow{w}$ resulting of the collision between a particle $\overleftarrow{w}$ and a particle g_e takes exactly one cell of delay compared to the incident particle $\overleftarrow{w}$ (cf figure 1 (b)). Thus, after i collisions, it takes exactly one cell of delay for the automaton 54^i (and thus separates two backgrounds) but the particle is not shifted of a whole cell of 54^{4j} and so there is a cell whose state is not a background one (containing the particle $\overleftarrow{w}$) what contradicts the assumption of existence of ι. $\square$

Corollary 5.1 *There exists an infinite number of distinct classes located below each automaton in* $54 \cup \nVdash$.

Proof 5.2 As we noticed above, for all $i \neq j$, $54^i \not\subseteq 54^j$ because $54^i \subseteq 54^j \Longrightarrow 54^{4i} \subseteq 54^{4j}$. Moreover for all k, 54^k has no fixed state (no configuration is insensitive to a collision with w particles). The theorem 4.1 thus ensures that all the automata of $54 \cup \nVdash$ are above an infinite number of classes. $\square$

Conclusion

The main result of this paper is the definition of the group G naturally introduced from the background pattern which gives information about 54 particles interactions. This method is general and can be applied (with more or less results) to any unidimensional cellular automata when backgrounds naturally appear in their space-time diagrams. We can for instance completely understand the rule 184 by this method, but it gives few information on 110 which has two backgrounds, many instable interactions and long time transitions.

This method gives a new filter on the rule 54. Many filters of the rule 54 have been designed (see for instance Boccara's paper) to better recognize the particles, but this one gives more information since it enables to understand their interactions.[2] More important, we can now rigorously prove negative results on the particles interactions and that the rule 54 has different behaviors at each scale. As a corollary we deduced a 3 state CA above an infinite number of classes in Rapaport's classification. Actually, to be above an infinite number of classes unfortunately does not mean to be algorithmically complex but rather to have a non self-similar sub-automaton. Anyway, Rapaport's classification seems to be a very interesting tool to study CAs states number optimization. In fact, we now better understand how few states CA may produce complex behaviors thanks to a background. The rule behavior is then completely linked to the background pattern.

This raises the following open questions:

- For a given Boccara's background, the number of possible particles and the diversity of their interactions is bounded by the group size and its minimal number of generators, it also seems interesting to know whether a combinatorial argument allows, from the states number, to give a bound on the "complexity" of all the possible groups.
- About the rule 54, two main questions are open: why this background almost always appears and none of the numerous others, and whether there are other particles in this background.
- The main question about 54 remains: is it Turing universal? Behind this question, are there Turing universal 2 states CA? Although we have now some tools to prove negative results on 54 behavior and that this behavior seems not complex enough to simulate an universal Turing machine, proving that 54 is not Turing universal is up to now impossible since there is no formal definition of Turing universality for CAs.

References

[1] F. Blanchard, E. Formenti, and P. Kurka. Cellular automata in the Cantor, Besicovitch and Weil topological spaces. *Complex Systems*, to appear 1999.

[2] N. Boccara, J. Nasser, and M. Roger. Particle-like structure and their interactions in spatio-temporal patterns generated by one-dimensional deterministic cellular-automata rules. *Physical Review A*, 44:866–875, 1991.

[3] G. Cattaneo, E. Formenti, L. Margara, and G. Mauri. Transformation of the one-dimensional cellular automata rule space. *Parallel Computing*, 23:1593–1611, 1997.

[4] G. Cattaneo, E. Formenti, L. Margara, and G. Mauri. On the dynamical behavior of chaotic cellular automata. *Theoretical Computer Science*, 217:31–51, 1999.

[5] J. Mazoyer and I. Rapaport. Additive cellular automata on $\mathbb{Z}_p$, and the bottom of $(CA, \leq)$. In *Preceedings of MFCS'98*, volume 1373, pages 834–843. Lecture Notes in Computer Science, Springer Verlag, 1998.

[6] J. Mazoyer and I. Rapaport. Inducing an order on cellular automata by a grouping operation. In *STACS'98*, volume 1373, pages 116–127. Lecture Notes in Computer Science, 1998.

[7] S. Wolfram. Universality and complexity in cellular automata. *Physica D*, 10:1–35, 1984.